Introduction to Deep Learning (I2DL)

Exercise 2: Math Recap

Overview

Linear Algebra

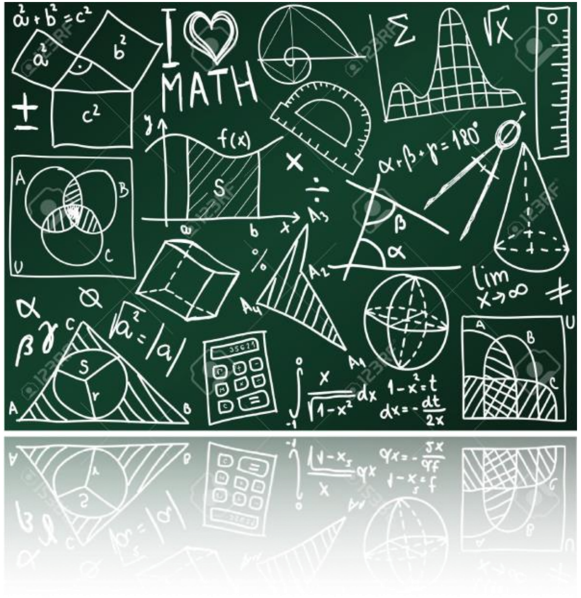
- Vectors and matrices
- Basic operations on matrices & vectors
- Tensors
- Norms, Loss functions

Calculus

- Scalar derivatives
- Gradient
- Jacobian Matrix
- Chain Rule

Probability Theory

- Probability space
- Random variables
- PMF, PDF, CDF
- Mean, variance
- Standard probability distributions



Linear Algebra

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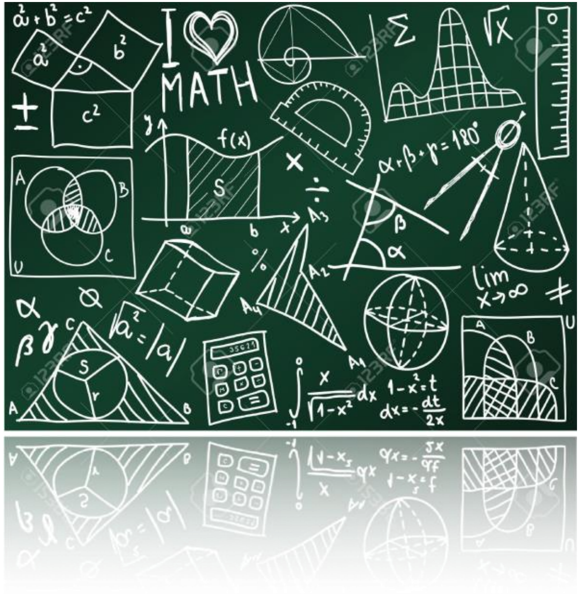


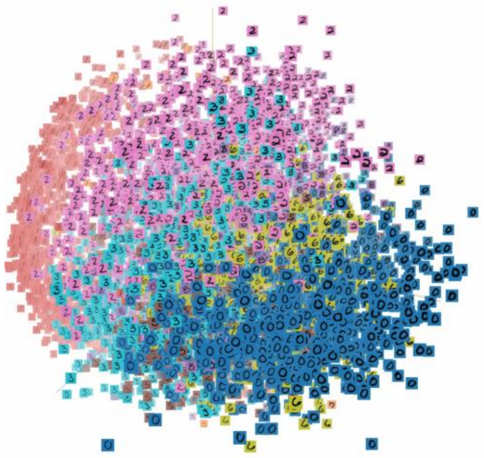
Basic Notation

- **Vector:** We call an element of $\mathbb{R}^n$ a vector with entries.
- **Elements of a vector:** The th element of a vector $v \in \mathbb{R}^n$ is denoted by $v_i \in \mathbb{R}$.
- **Matrix:** We call an element of $\mathbb{R}^{n \times m}$ a matrix with rows and columns.
- **Elements of a matrix:** For $A \in \mathbb{R}^{n \times m}$, we denote the element at the th row and jth column by $A_{ij} \in \mathbb{R}$.
- **Transpose:** The transpose of a matrix results from “flipping” rows and columns. We denote the transpose of a matrix $A \in \mathbb{R}^{n \times m}$ by $A^T \in \mathbb{R}^{m \times n}$. Similarly, we use transposed vectors.

Vector

An n-dimensional vector describes an element in an n-dimensional space

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$



Vector Operations:

Addition	Subtraction	Scalar Multiplication	Dot Product

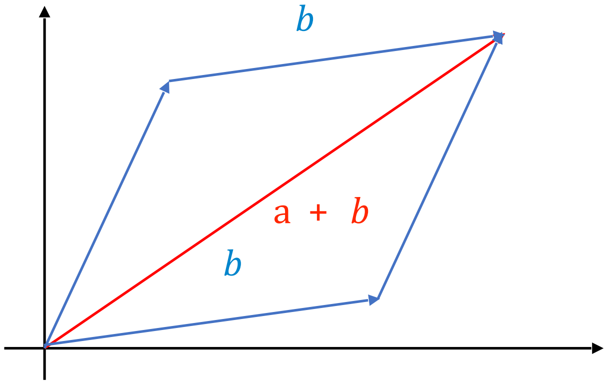
https://projector.tensorflow.org/

Vector Operations

Vector Operations: **Addition** | Subtraction | Scalar Multiplication | Dot Product

For $a, b \in \mathbb{R}^n$ we have

$$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} \in \mathbb{R}^n$$

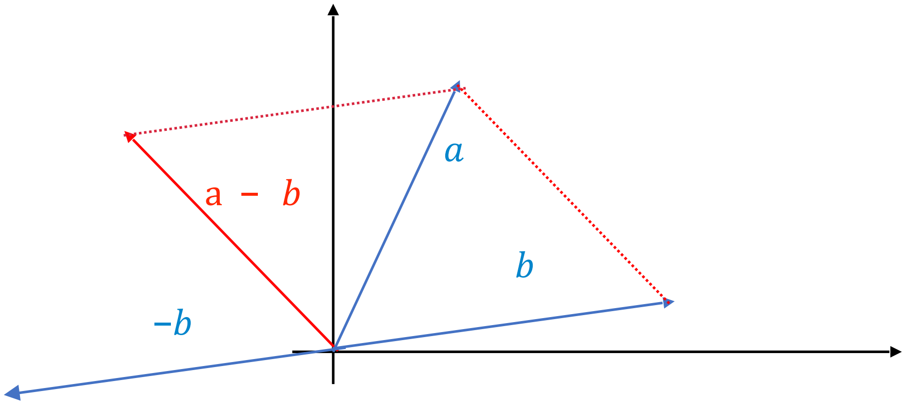


Vector Operations

Vector Operations: Addition | **Subtraction** | Scalar Multiplication | Dot Product

For $a, b \in \mathbb{R}^n$ we have

$$a - b = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{pmatrix} \in \mathbb{R}^n$$

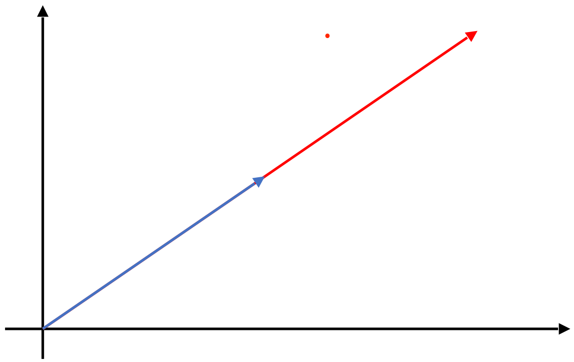


Vector Operations

Vector Operations: Addition | Subtraction | **Scalar Multiplication** | Dot Product

For $a \in \mathbb{R}^n, c \in \mathbb{R}$ we have

$$c \cdot a = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ \vdots \\ c \cdot a_n \end{pmatrix} \in \mathbb{R}^n$$



Vector Operations

Vector Operations:

Addition	Subtraction	Scalar Multiplication	**Dot Product**

Definition: For $a, b \in \mathbb{R}^n$, the dot product is defined as follows:

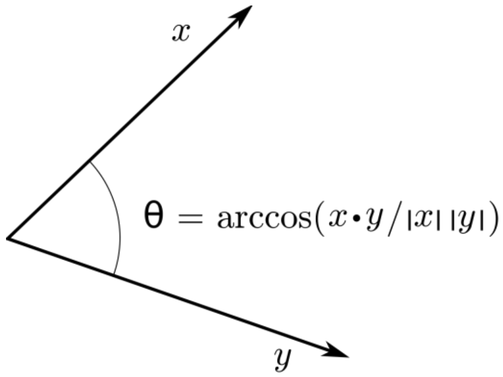
$$
\begin{aligned}
a \cdot b &= a^T \cdot b \\
&= a_1 \cdot b_1 + a_2 \cdot b_2 + \ldots + a_n \cdot b_n \\
&= \sum_{i=1}^{n} a_i \cdot b_i \in \mathbb{R}
\end{aligned}
$$

Vector Operations

Vector Operations: | Addition | Subtraction | Scalar Multiplication | **Dot Product**

Properties:

- Commutative: $a \cdot b = b \cdot$
- Geometric interpretation:
  $a \cdot b = \|a\| \cdot \|b\| \cdot \cos(\theta)$
- Orthogonality: Two non-zero vectors are orthogonal to each other $\Longleftrightarrow$ $a \cdot b = 0$



Vector Operations

Vector Operations: Addition | Subtraction | Scalar Multiplication | **Dot Product**

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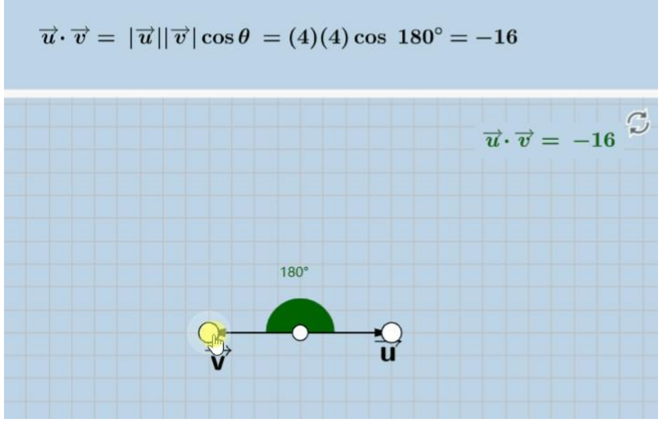


Matrix

A matrix $A \in \mathbb{R}^{n \times m}$ is denoted as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

Matrix Operations:

Matrix-vector Multiplication

Matrix-matrix Multiplication

Hadamard Product

Matrix

Matrix Operations: **Matrix-vector Multiplication** | Matrix-matrix Multiplication | Hadamard Product

- Multiplication of matrix with a vector is defined as follows:

$$\text{For } A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^{m}: A \cdot b = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_1 + a_{12} \cdot b_2 + \dots + a_{1m} \cdot b_m \\ a_{21} \cdot b_1 + a_{22} \cdot b_2 + \dots + a_{2m} \cdot b_m \\ \vdots \\ a_{n1} \cdot b_1 + a_{n2} \cdot b_2 + \dots + a_{nm} \cdot b_m \end{pmatrix} \in \mathbb{R}^{n}$$

- **Attention:** The respective dimension have to fit, otherwise the multiplication is not well-defined.

$$\Longrightarrow \underbrace{A}_{n\times m} \cdot \underbrace{b}_{m\times 1} = \underbrace{c}_{n\times 1}$$

- **Example:** $A \in \mathbb{R}^{3\times 2}, b \in \mathbb{R}^{2}$ with $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 18 \\ 28 \end{pmatrix}$

Matrix Operations

Matrix Operations: | Matrix-vector Multiplication | **Matrix-matrix Multiplication** | Hadamard Product

- Similar, the multiplication of two matrices with each other is defined as follows:
  For $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times l}$ we have

$$A \cdot B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1l} \\ b_{21} & b_{22} & \dots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{ml} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1l} \\ c_{21} & c_{22} & \dots & c_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nl} \end{pmatrix} \in \mathbb{R}^{n \times l} \text{ where}$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \dots + a_{im} \cdot b_{mj}$$

- Attention: Matrix Multiplication is in general not commutative, i.e. for two matrices $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times n}$ we have $A \cdot B \neq B \cdot A$

Matrix Operations

Matrix Operations: | Matrix-vector Multiplication | Matrix-matrix Multiplication | **Hadamard Product**

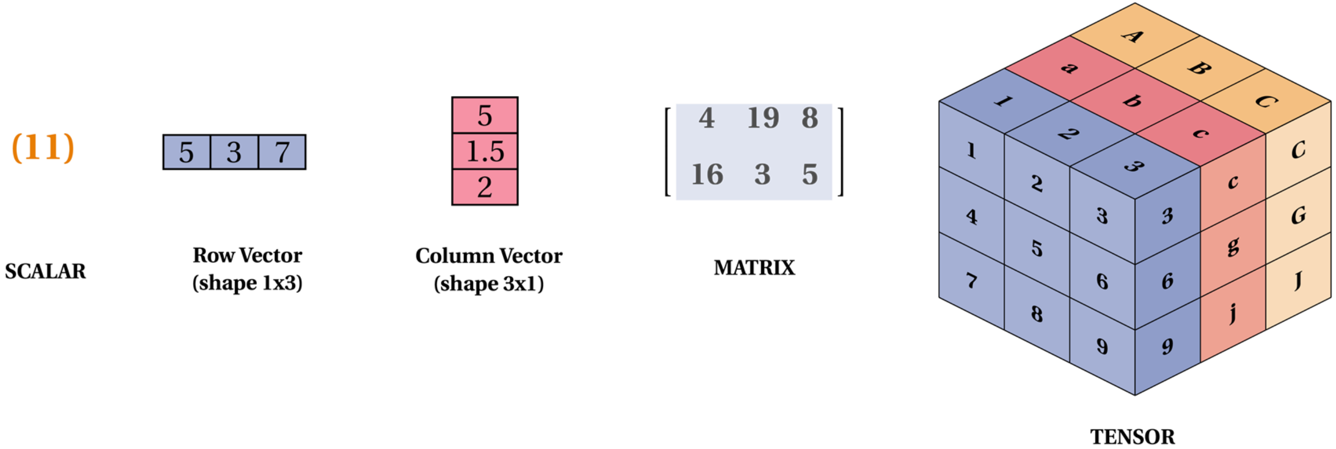
- The Hadamard product is the element wise product of two matrices. For two matrices of the same dimension $A, B \in \mathbb{R}^{n \times m}$ it is given by

$$A \odot B = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot b_{11} & \cdots & a_{1m} \cdot b_{1m} \\ a_{21} \cdot b_{21} & \cdots & a_{2m} \cdot b_{2m} \\ \vdots & \ddots & \vdots \\ a_{n1} \cdot b_{n1} & \cdots & a_{nm} \cdot b_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

For all matrix operations, it is important to check the dimensions!

Tensor

- Definition: A tensor is a multidimensional array and a generalization of the concepts of a vector and a matrix.

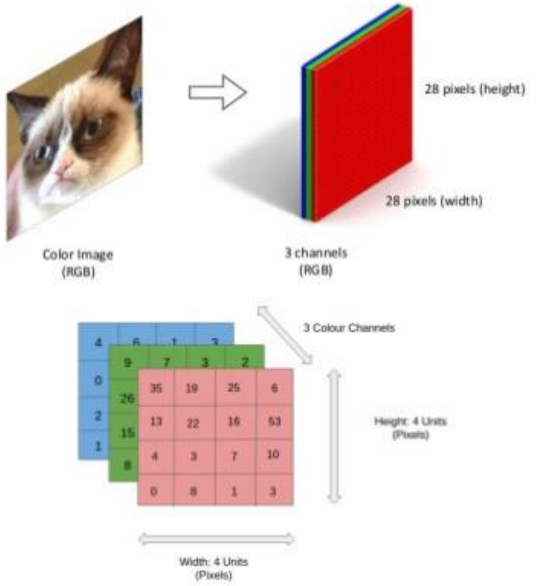


Tensors in Computer Vision

color image is 3rd-order tensor

Tensors are used to represent RGB images.

$$H \times W \times RGB$$



Source: https://www.slideshare.net/BertonEarnshaw/a-brief-survey-of-tensors

Norm

- **Norm:** measure of the "length" of a vector
- **Definition:** A norm is a non-negative function $\| \cdot \| : V \to \mathbb{R}$ which is defined by the following the properties for elements $v, w \in V$:
   1. Triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$
   2. $\|a \cdot v\| = a \cdot \|v\|$ for a scalar
   3. $\|v\| = 0$ if and only if $= 0$

  (* is a vector space over a field $\mathbb{F}$; in our case we have $= \mathbb{R}^n$)
- **Remark:** Every such function defines a norm on the vector space.
- **Examples:** L1-norm, L2-norm

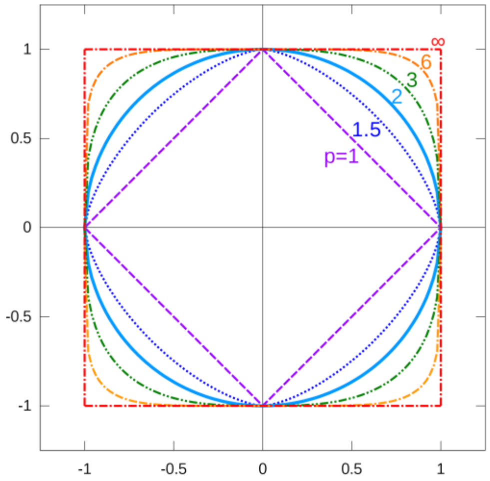
L1-Norm

- **Norm:** measure of the "length" of a vector
- **L1-Norm:** We denote the L1-norm with $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ such that for a vector $v = (v_1, v_2, \dots, v_n)$

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

- **Example:** Let $v = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \in \mathbb{R}^3$, then

$$\|v\|_1 = (1 + 3 + 2) = 6$$



https://en.wikipedia.org/wiki/Lp_space

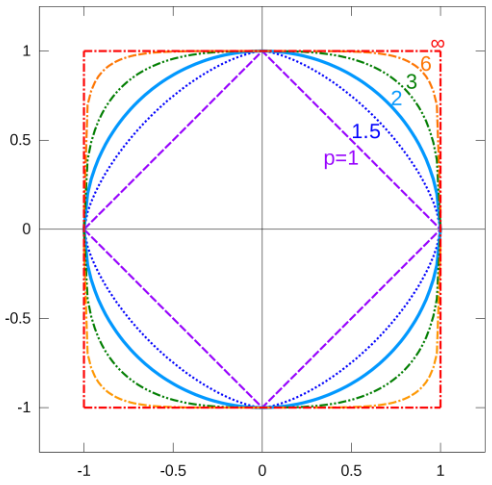
L2-Norm

- **Norm:** measure of the "length" of a vector
- **L2-Norm:** We denote the L2-norm with $\| \cdot \|_2 : \mathbb{R}^n \to \mathbb{R}$ such that for a vector $v = (v_1, v_2, \ldots, v_n)$

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} (v_i)^2}$$

- **Example:** Let $v = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \in \mathbb{R}^3$, then

$$\|v\|_2 = \sqrt{(1^2 + (-3)^2 + 2^2)} = \sqrt{14}$$



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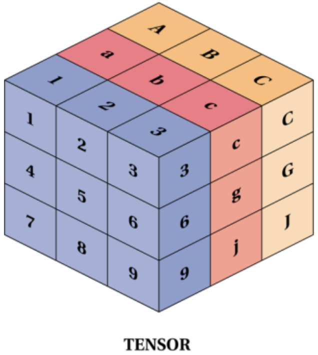
Loss functions

- A loss function is a function that takes as input two vectors and as output measures the distance between these two
  - uses a norm to measure the distance
  - L1-Loss uses the L1-norm, L2-Loss uses the L2-norm
- **L1-Loss:** The L1-Loss between two vectors $v, w \in \mathbb{R}^n$ is defined as $L_1(v, w) = \|v - w\|_1 = \sum_{i=1}^{n} |v_i - w_i$
- **L2-Loss:** The L2-Loss between two vectors $v, w \in \mathbb{R}^n$ is defined as

$$L_2(v, w) = \|v - w\|_2 = \sqrt{(v_1 - w_1)^2 + \ldots + (v_n - w_n)^2}$$

Outlook



Neural Network

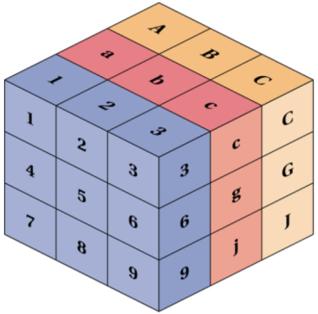
17% cat
13% dog
70% deer

Loss

[(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)](http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook

→



TENSOR

→

$$W \cdot x$$

$$W \in \mathbb{R}^{n \times m}$$

→

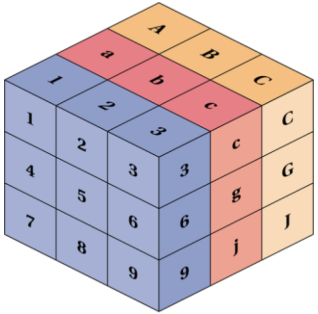
17% cat
13% dog
70% deer

→ Loss

The elements of the matrix   are called weights and they determine the prediction of our network.

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook



TENSOR

$$W \cdot x$$

$$W \in \mathbb{R}^{n \times m}$$

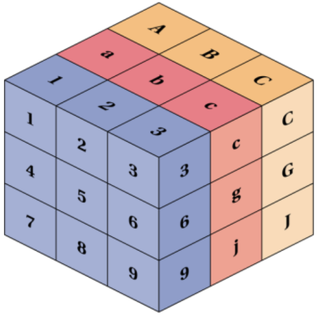
17% cat
13% dog
70% deer

Loss

How can we get an accurate matrix to minimize the loss?

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Outlook



TENSOR

$$W \cdot x$$

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17% cat
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Loss

Gradient Descent: Method to approximate the best values for the weights

(http://www.isfpga.org/fpga2017/slides/D2_S1_02.pdf)

Calculus

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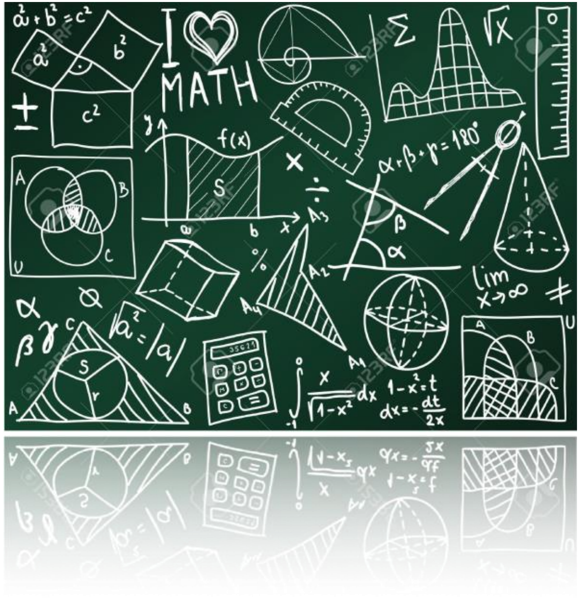


Derivatives

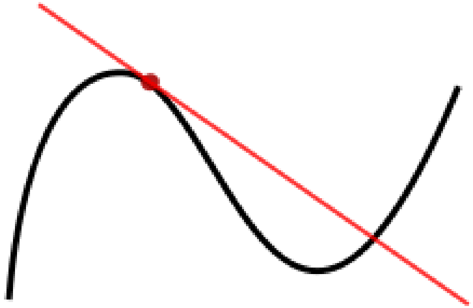
- **Well known:** Scalar derivatives, i.e. derivatives of functions $f : \mathbb{R} \rightarrow \mathbb{R}$
- **Matrix calculus:** Extension of calculus to higher dimensional setting, i.e. functions like $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $f : \mathbb{R} \rightarrow \mathbb{R}^n$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ for $n, m \in \mathbb{N}$
- Actual calculus we use is relatively trivial, but the notation can often make things look much more difficult than they are.

Overview

Setting	Derivative	Notation
$f : \mathbb{R} \to \mathbb{R}$	Scalar derivative	$f'(x)$
$f : \mathbb{R}^n \to \mathbb{R}$	Gradient	$\nabla f(x)$
$f : \mathbb{R}^{n \times m} \to \mathbb{R}$	Gradient	$\nabla f(x)$
$f : \mathbb{R}^n \to \mathbb{R}^m$	Jacobian	J_f

Scalar derivatives

- **Setting:** $f : \mathbb{R} \to \mathbb{R}$
- **Notation:** $f'(x)$ or $\dfrac{\mathrm{d}f}{\mathrm{d}x}$
- **Derivative:** Derivative of a function at a chosen input value is the slope of the tangent line to the graph of the function at that point.



https://en.wikipedia.org/wiki/Derivative#/media/File:Tangent_to_a_curve.svg

Derivation Rules

Common functions	Derivative
$f(x) = c$ for $c \in \mathbb{R}$	$f'(x) = 0$
$f(x) = x$	$f'(x) = 1$
$f(x) = x^n$ for $n \in \mathbb{N}$	$f'(x) = n \cdot x^{n-1}$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = sin(x)$	$f'(x) = cos(x)$
$f(x) = cos(x)$	$f'(x) = -sin(x)$

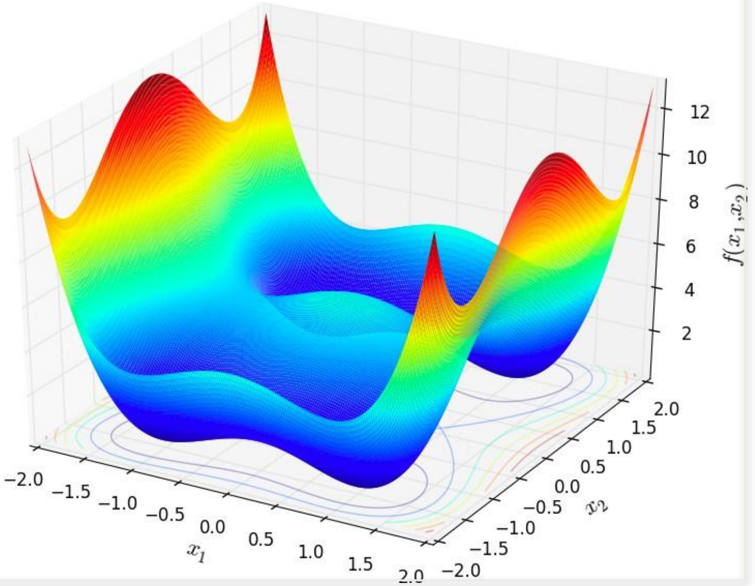
Derivation Rules

Rule	Function	Derivative
Sum rule	$f(x) + g(x)$	$f'(x) + g'(x)$
Difference rule	$f(x) - g(x)$	$f'(x) - g'(x)$
Multiplication by constant	$c \cdot f(x)$	$c \cdot f'(x)$
Product rule	$f(x) \cdot g(x)$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
Chain rule	$f(g(x))$	$f'(g(x)) \cdot g'(x)$

Multivariate functions $f: \mathbb{R}^n \to \mathbb{R}$

Multivariate Function

$$f: \mathbb{R}^n \to \mathbb{R}$$



Gradient

$$\nabla f: \mathbb{R}^n \to \mathbb{R}^n$$

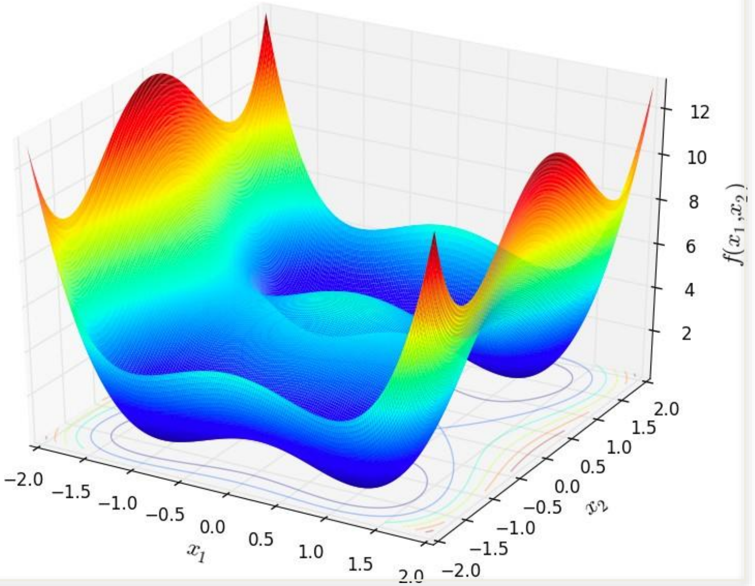
Partial derivative

$$\nabla f: x \to \nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

Multivariate functions $f: \mathbb{R}^{n\times m} \rightarrow \mathbb{R}$

Multivariate Function

$$f: \mathbb{R}^{n\times m} \rightarrow \mathbb{R}$$

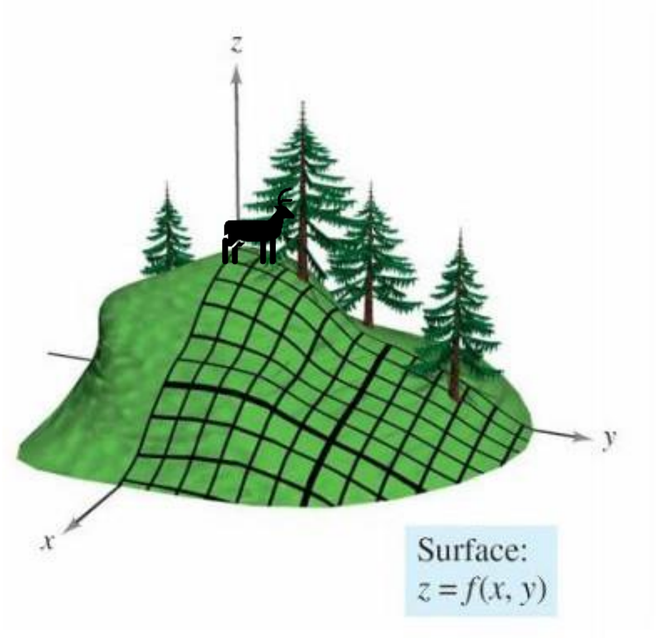


Gradient

$$\nabla f: \mathbb{R}^{n\times m} \rightarrow \mathbb{R}^{n\times m}$$

$$\nabla f : x \rightarrow \nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_{11}} & \frac{\partial f(x)}{\partial x_{12}} & \cdots & \frac{\partial f(x)}{\partial x_{1m}} \\ \frac{\partial f(x)}{\partial x_{21}} & \frac{\partial f(x)}{\partial x_{22}} & \cdots & \frac{\partial f(x)}{\partial x_{2m}} \\ \vdots & & & \\ \frac{\partial f(x)}{\partial x_{n1}} & \frac{\partial f(x)}{\partial x_{n2}} & \cdots & \frac{\partial f(x)}{\partial x_{nm}} \end{pmatrix}$$

Gradient – Example 1



$$f(x,y) = 3x^2y \qquad \nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right]$$

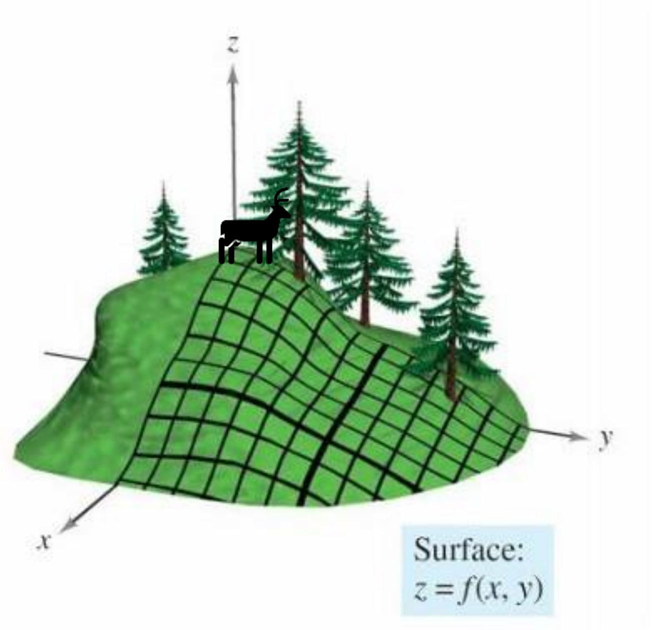
$$\frac{\partial}{\partial x} 3yx^2 = 3y \frac{\partial}{\partial x} x^2 = 3y2x = 6yx$$

$$\frac{\partial}{\partial y} 3x^2y = 3x^2 \frac{\partial}{\partial y} y = 3x^2 \frac{\partial y}{\partial y} = 3x^2 \times 1 = 3x^2$$

$$\nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right] = [6yx,\ 3x^2]$$

https://www.zhihu.com/question/36301367

Gradient – Example 2



$$g(x, y) = 2x + y^8$$

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial 2x}{\partial x} + \frac{\partial y^8}{\partial x} = 2\frac{\partial x}{\partial x} + 0 = 2 \times 1 = 2$$

$$\frac{\partial g(x,y)}{\partial y} = \frac{\partial 2x}{\partial y} + \frac{\partial y^8}{\partial y} = 0 + 8y^7 = 8y^7$$

$$\nabla g(x, y) = [2, 8y^7]$$

https://www.zhihu.com/question/36301367

Vector-valued functions

Vector-Valued function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f : x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \rightarrow \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

Jacobian Matrix

$$J_f : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times n}$$

$$x \rightarrow J_f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

Jacobian Matrix – Example 3

Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}$ where $f_1(x, y) = 3x^2 y$ and $f_2(x, y) = 2x + y^8$.

Calculate Jacobian matrix:

$$J_f(x) = \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 6xy & 3x^2 \\ 2 & 8y^7 \end{pmatrix}$$

Single Variable Chain Rule

Setting: We are given the function h(x) = f(g(x)).

Task: Compute the derivative of this function with chain rule.

1. **Introduce the intermediate variable:** Let $u = g(x)$ be the intermediate variable.
2. **Compute individual derivatives:** $\frac{\mathrm{d}f}{\mathrm{d}u}$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}$
3. **Chain rule:** $\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$
4. **Substitute intermediate variables back**

Single Variable Chain Rule: Example

Example: Let $h(x) = sin(x^2)$.

Task: Compute the derivative of this function with chain rule.

Observation: Here, $h(x) = f(g(x))$ with $f(x) = sin(x)$ and $g(x) = x^2$.

1. **Introduce the intermediate variable:** Let $u = x^2$ be the intermediate variable.

2. Compute individual derivatives: $\frac{\mathrm{d}f}{\mathrm{d}u} = cos(u)$ and $\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

3. Chain rule: $\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = cos(u) \cdot 2x$

4. Substitute intermediate variables back: $\frac{\mathrm{d}h}{\mathrm{d}x} = cos(u) \cdot 2x = cos(x^2) \cdot 2x$

Total Derivative Chain Rule

General Formalism:

$$\frac{\partial f(x, u_1(x), \dots, u_n(x))}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x} + \dots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x}$$

$$= \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial x}$$

References

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Probability Theory

Overview

Linear Algebra

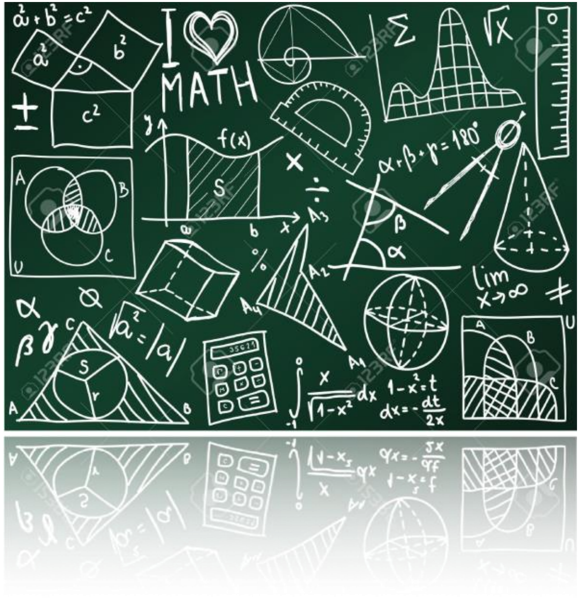
- Vectors and matrices
- Basic operations on matrices & vectors
- Tensors
- Norm & Loss functions

Calculus

- Scalar derivatives
- Gradient
- Jacobian Matrix
- Chain Rule

Probability Theory

- Probability space
- Random variables
- PMF, PDF, CDF
- Mean, variance
- Standard probability distributions



Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

A probability space consist of three elements $(\Omega, \mathcal{F}, \mathbb{P})$:

- **Sample space Ω:** The set of all outcomes of a random experiment.
- **Event Space $\mathcal{F}$:** A set whose elements $A \in \mathcal{F}$ (called events) are subsets of Ω.
- **Probability measure $\mathbb{P}$:** A function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ that satisfies the following three properties:
    1. $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$
    2. $\mathbb{P}(\Omega) = 1$
    3. $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ for $n \in \mathbb{N}$ and disjoint events $A_1, A_2, \ldots A_n \in \mathcal{F}$

The probability space provides a formal model of a random experiment.

Probability space: Example

A probability space consists of three elements: $(\Omega, \mathcal{F}, \mathbb{P})$

- **Sample space Ω:** The set of all outcomes of a random experiment.
- **Event Space $\mathcal{F}$:** A set whose elements $A \in \mathcal{F}$ (called events) are subsets of Ω.
- **Probability measure $\mathbb{P}$:** A function $\mathbb{P}: \mathcal{F} \to [0, 1]$ that satisfies the following three properties: (…)

Example: Tossing a six-sided die

- **Sample space:** $\Omega = \{1,2,3,4,5,6\}$
- **Event space:** $\mathcal{F}_1 = \{\emptyset, \Omega\}$, $\mathcal{F}_2 = \mathcal{P}(\Omega)$,
  $\mathcal{F}_3 = \{\emptyset, A_1 = \{1,3,5\}, A_2 = \{2,4,6\}, \Omega = \{1,2,3,4,5,6\}\}$
- **Probability measure $\mathbb{P}: \mathcal{F} \to \mathbb{R}$** with $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ and in the case of $\mathcal{F}_3$ we know that $\mathbb{P}(A_1) + \mathbb{P}(A_2) = 1$.
- **Example event space $\mathcal{F}_3$:** Possible probability measure are
  1. $\mathbb{P}_1(A_1) = \frac{1}{2} = \mathbb{P}_1(A_2)$
  2. $\mathbb{P}_2(A_1) = \frac{1}{4}$ and $\mathbb{P}_2(A_2) = \frac{3}{4}$.

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Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e.

$$X : \Omega \to \mathbb{R}.$$

- We distinguish between **discrete** and **continuous** random variables.

Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: Tossing a fair six-sided die

- **Underlying experiment:** $\Omega = \{1,2,3,4,5,6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}(\{x\}) = \frac{1}{6} \forall x \in \Omega$
- **Random variable** : Number that appears on the die, $X : \Omega \to \{1,2,3,4,5,6\}$
  $\Longrightarrow$ discrete random variable
- **Example:** One element in Ω is $= 4$. Then $X(\omega) = 4$.
- **Probability measure** $\mathbb{P}$**:**

$$\mathbb{P}(X = 4) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = \omega = 4\}) = \mathbb{P}(\{4\}) = \frac{1}{6}$$

discrete

https://images.app.goo.gl/GMfyoXi9trZgqecQ8

Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: Flipping a fair coin two times

- **Underlying experiment:** $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$,

  $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega\}) = \frac{1}{4}\ \forall \omega \in \Omega$

- **Random variable :** number of heads that appeared in the two flips, $X : \Omega \to \{0,1,2\}$

  $\Longrightarrow$ discrete random variable

- **Example:** One element in Ω is $\omega = (T,H)$. Then $X(\omega) = 1$.

- **Probability measure $\mathbb{P}$:**

$$\mathbb{P}(X = 1) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 1\}) = \mathbb{P}(\{(H,T),(T,H)\}) = \frac{1}{2}$$

discrete

Random variable

- A random variable is a function defined on the probability space which maps from the sample space to the real numbers, i.e. $X : \Omega \to \mathbb{R}$.

Example: radioactive decay

- **Underlying experiment:** $\Omega = \mathbb{R}_{\geq 0}, \mathcal{F} = \mathcal{B}(\Omega), \mathbb{P} =$ is the Lebesgue measure
- **Random variable :** indicating amount of time that it takes for a radioactive particle to decay, $X : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \implies$ continuous random variable
- **Probability measure $\mathbb{P}$:** is defined on the set of events $\mathcal{F}$ and is now used for random variables as follows:
$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(\{\omega \in \Omega : a \leq X(\omega) \leq b\})$

Continuous

Probability measures

$\Longrightarrow$ specify the probability measures with alternative functions (CDF, PDF and PMF)

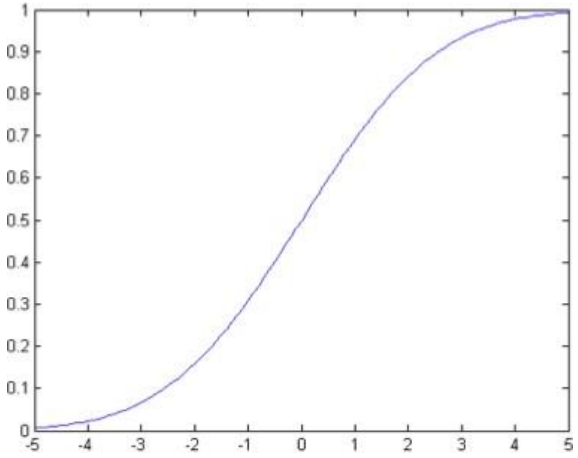
Random Variable		
Discrete	Cumulative distribution function (CDF) $F_X(x) = \mathbb{P}(X \leq x)$	Probability mass function (PMF) $p_X(x) = \mathbb{P}(X = x)$
Continuous	Cumulative distribution function (CDF) $F_X(x) = \mathbb{P}(X \leq x)$	Probability distribution function (PDF)

Cumulative Distribution Function

- A **cumulative distribution function** (CDF) of a random variable is a function $F_X : \mathbb{R} \rightarrow [0,1]$ which is defined as

$$F_X(x) = \mathbb{P}(X \leq x)$$

- **Properties:** Per definition, it satisfies the following properties:
  1. $0 \leq F_X(x) \leq 1$
  2. $\lim_{x \rightarrow -\infty} F_X(x) = 0$
  3. $\lim_{x \rightarrow \infty} F_X(x) = 1$
  4. $\forall x \leq y \implies F_X(x) \leq F_X(y)$



A sample CDF

[Figure: http://cs229.stanford.edu/section/cs229-prob.pdf]

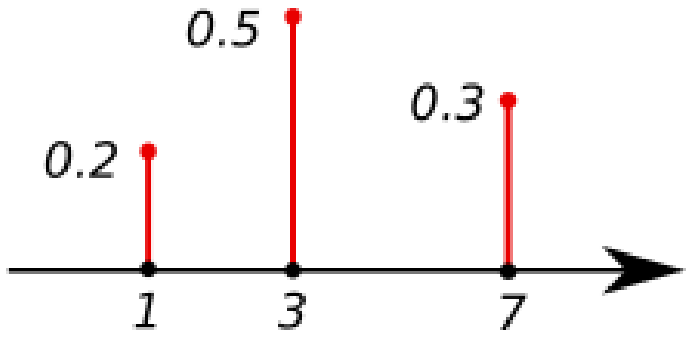
Discrete Case: Probability Mass Function

- The **probability mass function** of a random variable is a function $p_X : \Omega \rightarrow \mathbb{R}$ defined as

$$p_X(x) = \mathbb{P}(X = x)$$

- **Properties:** Again, we can derive some properties:

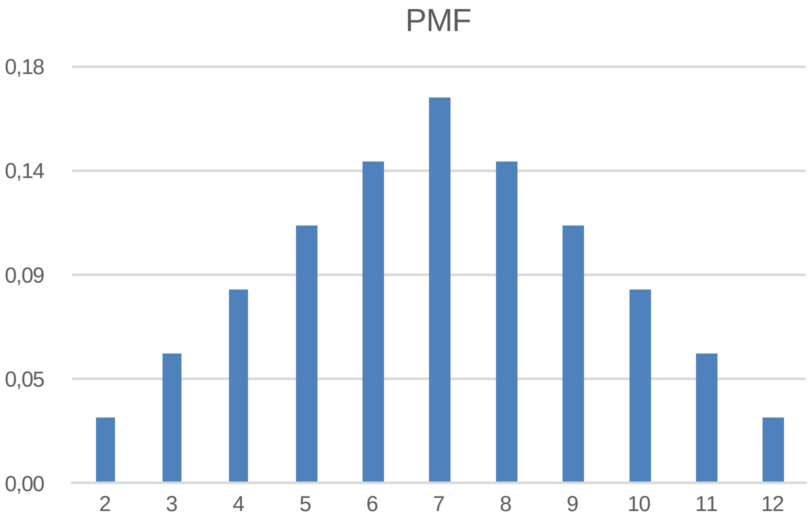
1. $0 \leq p_X(x) \leq 1$
2. $\sum_{x \in \Omega} p_X(x) = 1$

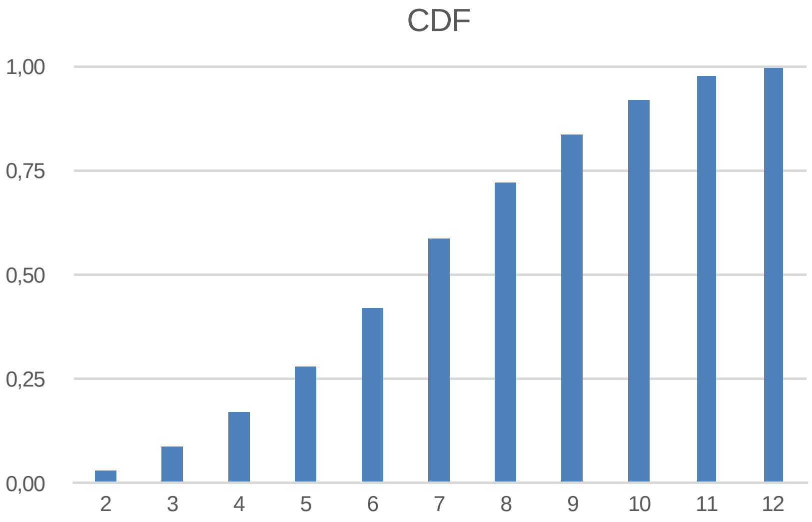


A sample PMF

[Figure: https://en.wikipedia.org/wiki/Probability_mass_function]

Discrete Example: Sum of 2 Dice Rolls





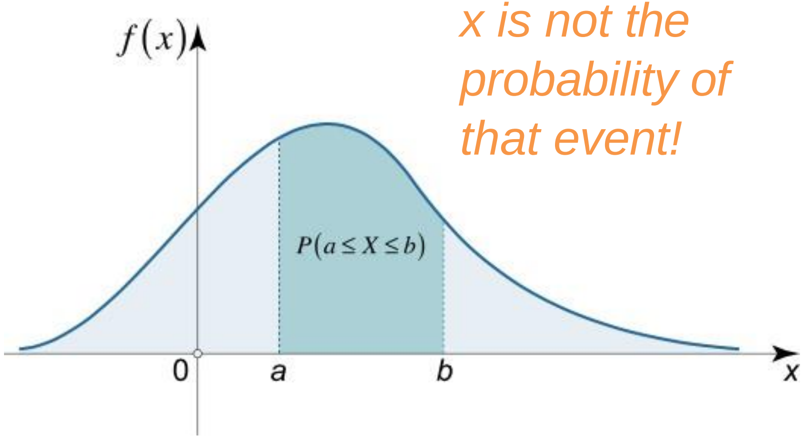
Continuous case: Probability Density Function

- **Continuous case:** For some continuous random variables, the CDF $F_X(x)$ is differentiable everywhere. Then we define the probability density function as the function $f_X(x) : \Omega \to \mathbb{R}$ with

$$f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}$$

Note: the value of a PDF at any given point x is not the probability of that event!

- **Properties:**
  1. $f_X(x) \geq 0$
  2. $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1$
  3. $\int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$



[Figure:https://www.math24.net/probability-density-function/]

Expectation of a random variable

- **Idea:** “weighted average” of the values that the random variable can take on
- **Discrete setting:** Assume that   is a discrete random variable with PMF $p_X(x)$. Then the expectation of   is given by

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x)$$

- **Continuous setting:** Assume that   is a continuous random variable with PDF $f_X(x)$. Then the expectation of   is given by

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \,\mathrm{d}x$$

Expectation: Example

- **Discrete setting:** Assume that is a discrete random variable with PMF $p_X(x)$. Then the expectation of is given by

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x)$$

Example: Tossing a six-sided die

$$\Omega = \{1,2,3,4,5,6\}$$

X: represents the outcome of the toss

$$p_X(x) = \mathbb{P}(X = x) = \frac{1}{6} \forall x \in \Omega$$

$$\mathbb{E}[X] = \sum_{x \in \Omega} x \cdot p_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

https://images.app.goo.gl/GMfyoXi9trZgqecQ8

Expectation of a random variable

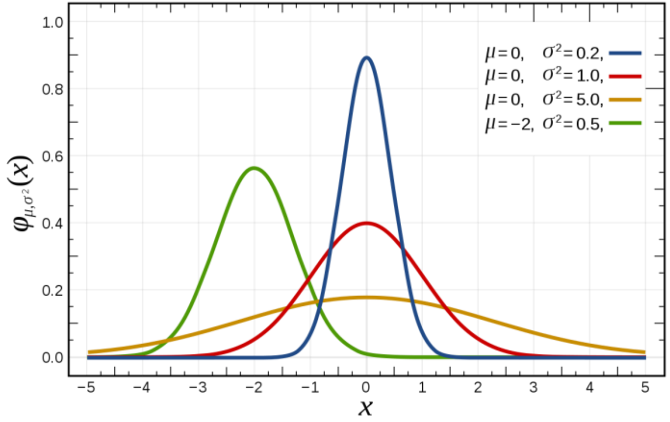
Properties: We encounter several important properties for the expectation, i.e.

1. $\mathbb{E}[a] = a$ for any constant $a \in \mathbb{R}$
2. Linearity: $\mathbb{E}[aX + bY] = a \cdot \mathbb{E}[X] + b \cdot \mathbb{E}[Y]$ for any constants $a, b \in \mathbb{R}$

Variance of a random variable

- **Idea:** The variance of a random variable is a measure how concentrated the distribution of a random variable is around its mean.
- **Definition:** The variance is defined as

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$



https://en.wikipedia.org/wiki/Normal_distribution

Variance of a random variable

Definition: The variance is defined as $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Example: Tossing a fair six-sided die

$\Omega = \{1,2,3,4,5,6\}$, X: represents the outcome of the toss

$$p_X(x) = \mathbb{P}(X = x) = \frac{1}{6}\ \forall x \in \Omega$$

$$\mathbb{E}[X] = 3.5,\ \mathbb{E}[X]^2 = 12\frac{1}{4}$$

$$\mathbb{E}[X^2] = \sum_{x\in\Omega} x^2 \cdot p_X(x) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \ldots + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = 15\frac{1}{6}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 15\frac{1}{6} - 12\frac{1}{4} = \frac{35}{12} \approx 2.91$$

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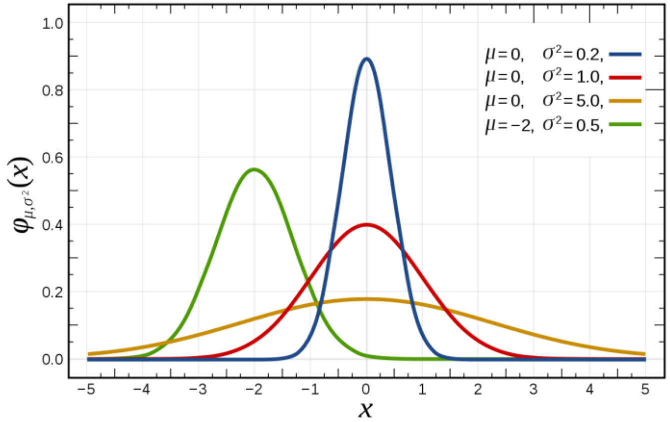
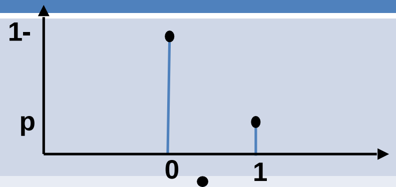
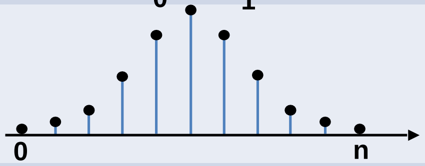
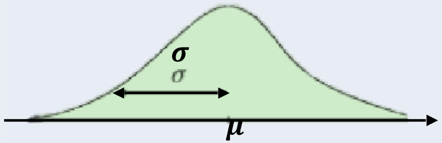
Variance of a random variable

- **Properties:** The variance has the following properties, i.e.
  1. $\mathrm{Var}(a) = 0$ for any constant $a \in \mathbb{R}$
  2. $\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X)$



https://en.wikipedia.org/wiki/Normal_distribution

Standard Probability Distributions

Distribution	Parameter & Notation	PDF or PMF	Mean	Variance	Illustration
Bernoulli distribution (Discrete)	$X \sim \text{Ber}(p)$ $0 \leq p \leq 1$	$p_X(k) = p^k(1-p)^{1-k}$	$\mathbb{E}[X] = p$	$\text{Var}(X) = p(1-p)$	
Binomial distribution (Discrete)	$X \sim \text{Bin}(n, p)$ $n \in \mathbb{N}, p \in [0,1]$	$p_X(k) = \binom{n}{k} p^k(1-p)^{n-k}$	$\mathbb{E}[X] = n \cdot p$	$\text{Var}(X) = np(1-p)$	
Uniform distribution (Continuous)	$X \sim U(a, b)$ $-\infty < a < b < \infty$	$f_X(x) = \begin{cases} \frac{1}{(b-a)} & x \in [a,b] \\ 0 & \text{else} \end{cases}$	$\mathbb{E}[X] = \frac{1}{2}(a+b)$	$\text{Var}(X) = \frac{1}{12}(b-a)^2$	
Normal distribution (Continuous)	$X \sim \mathcal{N}(\mu, \sigma^2)$ $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}_{\geq 0}$	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mathbb{E}[X] = \mu$	$\text{Var}(X) = \sigma^2$	

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  - Comprehensive Probability Review – **recommended!**
- https://stanford.edu/~shervine/teaching/cme-106/cheatsheet-probability
  - Quick Overview
- https://www.deeplearningbook.org/contents/prob.html
  - Another great resource. Also covers information theory basics.